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REMARKS ON UNIFORMITY IN HYPERELASTIC MATERIALS

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Abstract—The basic notions of material uniformity, originally developed by Noll and Wang, are presented using more traditional mathematical tools. Attention is focused on hyperelastic materials and new results are obtained pertaining to the arbitrariness of the strain energy level. The role of the isotropy group in allowing for the smooth extension of a material connection over the whole body is brought out.

1. INTRODUCTION

The notion of material uniformity within the framework of contemporary nonlinear continuum mechanics owes its origins to Noll[1]. This concept was generalized and further developed by Wang[2]. For a detailed historical account as well as a bibliography to related work we refer the reader to Bloom[3]. The contribution of Wang is particularly remarkable in that he was able to formulate the related physical concepts and developments in terms of modern global differential geometry, which relies on such abstract notions as fiber bundles and Lie groups. While rich in geometrical insight, these developments seem to lie beyond the grasp of the vast number of researchers and users of continuum mechanics. It is our opinion that this unfortunate circumstance has prevented a potentially useful theory from spreading and attaining its well deserved position in the study of the physics of continuous media.

The main objective of this paper is to present the basic notions of material uniformity in a manner more in keeping with traditional developments in continuum mechanics and thus make them accessible to workers in the field. Besides this shift in point of view, some new results are presented in this paper. Of these, we cite:

(i) We introduce the notion of curvewise smooth uniformity and develop criteria for it. In particular we introduce a differential material descriptor Γ_c, which plays a role akin to Wang's material connection[2].

(ii) We treat hyperelastic rather than elastic material response. Paradoxically, this serves to complicate the analysis due to the appearance of two scalar functions in addition to the strain energy density W; these are U, related to the arbitrariness of the energy level at points of the body and V, related to the fact that the isotropy groups for elastic stress and energy are in general not the same.

(iii) We bring out the role played by the isotropy group in contributing to the non-uniqueness of the material descriptor Γ_c. The results here are summarized in Propositions 2 and 4.† Moreover, we show in Proposition 3 how the non-uniqueness may be utilized to patch together regions of the body each of which separately enjoys smooth uniformity.

Section 2 introduces the concept of uniformity in the context of hyperelasticity. In Section 3 the local material symmetries of a uniform hyperelastic material are discussed and characterized in terms of a single *uniformized* or *basic* isotropy group. Section 4 presents the notion of curve-wise smooth uniformity which is then related in Section 6, to locally and globally smooth uniformity. Section 5 attempts to underline the role played by the isotropy group in

†In particular, it is important to mention that our eqn (6.6) can be shown essentially equivalent to Wang's field condition: [2], eqn (9.32).

allowing the freedom required to patch-up locally integrable descriptors into a single smooth, and in general non-integrable, material connection in the sense of Wang[2].

2. MATERIAL UNIFORMITY

Roughly speaking, a body $\mathcal{B}$ is materially uniform if all its points are made of the same material. Information concerning the material behaviour at a point X is obtained by mapping a small neighbourhood of that point into different configurations in physical Euclidean space E_3 and measuring the response (e.g. the stress tensor). It is customary to identify the body $\mathcal{B}$ with one of its configurations and to call deformation the mapping from this reference configuration to any arbitrary one. For a first-order (simple) elastic material, the response depends only on the gradient (first derivative) F of the deformation. In particular, we shall concentrate our attention on the hyper-elastic case, in which the response is completely described by a single scalar function W, the strain energy density. Thus, in general, the constitutive law for a hyperelastic body will be given as a function

$$W = W(F, X), \tag{2.1}$$

in which the gradient F is evaluated at the body point X.

To recognize whether or not 2 different points, X and Y, say, are made of the same material, it is obviously enough to check whether there exists a mapping between a neighbourhood $\mathcal{V}$ of Y and a neighbourhood $\mathcal{U}$ of X such that, after carrying $\mathcal{V}$ onto $\mathcal{U}$, the response of $\mathcal{V}$ to arbitrary subsequent deformation becomes identical to the response of $\mathcal{U}$. Since we are dealing with first-order materials, the mapping between $\mathcal{V}$ and $\mathcal{U}$ manifests itself only through its (non-singular) first derivative $P_X(Y)$, which is a linear isomorphism carrying vectors at Y onto vectors at X.

Physically (see Fig. 1), one "cuts" a small element around Y and one manages to squeeze it into a small hole around X, so that the mending patch at X functions exactly as the original element before the transplant. If such an operation is possible, one says that X and Y are materially isomorphic, and $P_X(Y)$ is a material isomorphism. In terms of the strain energy density we may express the condition of material isomorphism between X and Y as

$$W(F, X) = W[FP_X(Y), Y] + C, \tag{2.2}$$

where the constant C needs to be introduced since the energy reference level is arbitrary.

We note at this point that the material equivalence between X and Y could also have been formulated in terms of two isomorphisms $P(X)$ and $P(Y)$ between each of the points and the fixed standard translation space V_3 of Euclidean space E_3. Thus, X and Y are materially

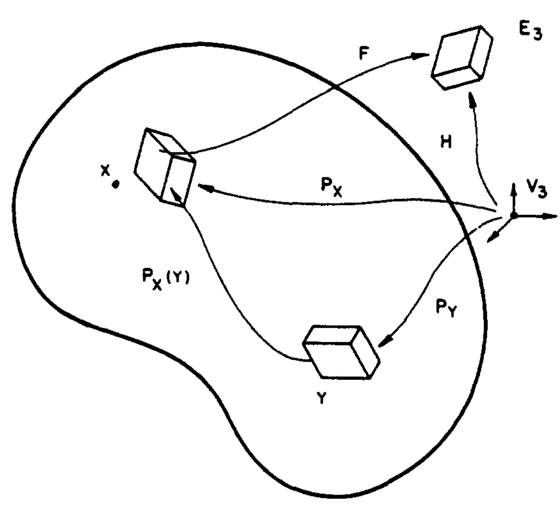

Fig. 1.

isomorphic if there exist two such isomorphisms such that

$$W(\boldsymbol{HP}(\boldsymbol{X}), \boldsymbol{X}) + C_1 = W(\boldsymbol{HP}(\boldsymbol{Y}), \boldsymbol{Y}) + C_2 = \hat{W}(\boldsymbol{H}), \tag{2.3}$$

where C_1, C_2 are constant and where now $\boldsymbol{H}$ represents an arbitrary non-singular linear mapping from V_3 onto the space of vectors ("tangent space") at an arbitrary point in physical space (Fig. 1). The connection between eqns (2.3) and (2.2) is given by

$$\boldsymbol{P}_X(\boldsymbol{Y}) = \boldsymbol{P}_X\boldsymbol{P}(\boldsymbol{Y}), \tag{2.4}$$

and

$$C = C_2 - C_1, \tag{2.5}$$

where the notation

$$\boldsymbol{P}_X = \boldsymbol{P}^{-1}(\boldsymbol{X}), \tag{2.6}$$

is consistent with the obvious statement

$$\boldsymbol{P}_Y(\boldsymbol{X}) = \boldsymbol{P}_X^{-1}(\boldsymbol{Y}). \tag{2.7}$$

We are now in a position to define a uniform hyperelastic body as one for which all points are pairwise materially isomorphic. Using the concepts and notation embodied in eqn (2.3) we state:

Definition 1

A hyperelastic body $\mathscr{B}$ with constitutive law

$$W = W(\boldsymbol{F}, \boldsymbol{X}) \tag{2.1}$$

is said to be *uniform* if there exists a field of linear isomorphisms $\boldsymbol{P}_X$

$$\boldsymbol{P}_X : V_3 \to T_X\mathscr{B}, \tag{2.8}$$

a field of scalars U

$$U : \mathscr{B} \to R, \tag{2.9}$$

and a scalar valued 2-point tensor function $\hat{W}$ such that

$$W = W(\boldsymbol{F}, \boldsymbol{X}) = \hat{W}(\boldsymbol{FP}_X) + U(\boldsymbol{X}). \tag{2.10}$$

In eqn (2.8) $T_X\mathscr{B}$ denotes the space of tangent vectors at $\boldsymbol{X} \in \mathscr{B}$ and in eqn (2.9) R denotes the real line. The word field is used loosely, since at this point we do not require any smoothness condition.

3. UNIFORM MATERIAL SYMMETRIES

Let

$$\boldsymbol{G}_X : T_X\mathscr{B} \to T_X\mathscr{B} \tag{3.1}$$

be a material symmetry for the stress response at $\boldsymbol{X}$, viz.

$$W(\boldsymbol{F}, \boldsymbol{X}) = W(\boldsymbol{FG}_X, \boldsymbol{X}) + V_X(\boldsymbol{G}_X) \tag{3.2}$$

for all F, where $V_X(G_X)$ is a constant, and thus leaves the stress response unaffected ([4], p. 310). It follows immediately that if P_X is an isomorphism used as in (2.8) to define uniformity, it can be replaced by the isomorphism $G_X P_X$. Indeed, combining eqns (2.10) with (3.2) we obtain

$$W(F,X) = W(FG_X P_X) + U(X) + V_X(G_X), \tag{3.3}$$

which is again of the form (2.10). Moreover, it is not difficult to see that the tensor

$$G_Y = P_Y P_X^{-1} G_X P_X P_Y^{-1}, \tag{3.4}$$

which by virtue of (2.4) can also be written as

$$G_Y = P_Y(X) G_X P_X(Y) \tag{3.5}$$

is a material symmetry at Y. Indeed, using eqns (3.4), (2.10) and (3.3) repeatedly we get

$$\begin{aligned} W(FG_Y, Y) &= W(FP_Y P_X^{-1} G_X P_Y^{-1}, Y) \\ &= \hat{W}(FP_Y P_X^{-1} G_X P_X) + U(Y) \\ &= W(FP_Y P_X^{-1}, X) + U(Y) - U(X) - V_X(G_X) \\ &= \hat{W}(FP_Y) + U(Y) - V_X(G_X) \\ &= W(F, Y) - V_X(G_X), \end{aligned} \tag{3.6}$$

which incidentally shows that V_X is the same at all points X for corresponding elements in the isotropy group. In uniform bodies, all local symmetries are connected in this way, i.e. by eqn (3.4). Notice that the composition

$$G = P_X^{-1} G_X P_X = P_Y^{-1} G_Y P_Y \tag{3.7}$$

is, by virtue of (3.4) independent of position in the body. Also, by a trivial use of eqn (2.10) it follows that

$$\hat{W}(H) = \hat{W}(HG) + V(G), \tag{3.8}$$

for all H. From here it follows that in a uniform body all material symmetries are typified by the *basic* group of isotropy $\mathscr{G}$ of the *uniformized* strain energy density $\hat{W}$.

In eqn (3.8) $V(G)$ is the common value of $V_X(G_X)$ for all X and corresponding G_X.

Let G_1 and G_2 belong to $\mathscr{G}$. Then a repeated use of eqn (3.8), viz.

$$\begin{aligned} \hat{W}(H) &= \hat{W}(HG_1) + V(G_1) \\ &= \hat{W}(HG_1 G_2) + V(G_2) + V(G_1), \end{aligned} \tag{3.9}$$

shows that

$$V(G_1 G_2) = V(G_1) + V(G_2). \tag{3.10}$$

When V depends smoothly on $\mathscr{G}$ we may differentiate with respect to G_2 and then set $G_2 = 1$ to obtain

$$G_1{}^T \left\{ \frac{\partial V}{\partial G}\bigg|_{G=G_1} \right\} = \frac{\partial V}{\partial G}\bigg|_{G=1} \tag{3.11}$$

We also note the obvious result

$$V(1) = 0 \tag{3.12}$$

which follows, for instance, from (3.10) with $G_1 = G_2 = 1$.

A useful tool in some manipulations pertaining to the analysis of uniform bodies is that of *uniformity basis*. Let $e_i (i = 1, 2, 3)$ be a (usually orthonormal) basis in the standard translation space V_3 of E_3, and let P_X be the field of isomorphisms (2.8) used in the definition of uniformity. Then the field of local bases

$$f_{Xi} = P_X e_i \tag{3.13}$$

defines a uniformity basis in $\mathcal{B}$.

Now, let f_{Xi} be a uniformity basis in $\mathcal{B}$ induced by a basis e_i in V_3. In such a basis the components of the two-point tensors P_X are given simply by the unit matrix, viz

$$P_X = \delta_j^i f_{Xi} \otimes e^j. \tag{3.14}$$

Indeed, for all $v = v^i e_i \in V_3$,

$$P_X v = P_X(v^i e_i) = v^i P_X e_i = v^i f_{Xi}. \tag{3.15}$$

It follows now that the components of G and G_X, connected by eqn (3.7), in their respective bases e_i and f_{Xi}, turn out to be the same. Thus, when using a uniformity basis, the material symmetries of a uniform body are represented by a single matrix group, namely, the component representation of the group $\mathcal{G}$ of $\hat{W}$ in any fixed basis e_i.

4. CURVE-WISE SMOOTH UNIFORMITY

As remarked in Section 2, so far we have not imposed any smoothness conditions on the field of material isomorphisms. As a first step in that direction we introduce now the idea of smooth material uniformity along a given curve $c(s)$ by demanding that a field of material isomorphisms $P_X(s)$ exist along the curve, such that $P_X(s)$ is a smooth function of the parameter s. More precisely, let

$$c : [-\epsilon, \epsilon] \to \mathcal{B}, (\epsilon > 0), \tag{4.1}$$

be given such that

$$c(0) = X_0, \tag{4.2}$$

and let $\mathcal{B}$ be materially uniform.

Definition 2

We say that $\mathcal{B}$ is smoothly uniform at X_0 along c if a field P_X of uniformity (2.8) can be found such that the composition

$$P_c = P_X \circ c \tag{4.3}$$

is smooth.

We now seek to obtain a differential condition for this curvewise smooth uniformity. To this end we combine eqn (2.10) and (4.3) to yield

$$W(F, c(s)) = \hat{W}(FP_c(s)) + U(c(s)), \tag{4.4}$$

and we differentiate with respect to the curve parameter s, viz

$$\dot{W} = \mathrm{tr}\left[\left(\frac{\partial \hat{W}}{\partial FP_c}\right)^T F\dot{P}_c\right] + \dot{U}, \tag{4.5}$$

where $\cdot \equiv d/ds$, and with respect to F, viz

$$\frac{\partial W}{\partial F} = \frac{\partial W}{\partial F P_c} P_c^T. \tag{4.6}$$

Combining eqns (4.5) and (4.6) we obtain

$$\dot{W} = \mathrm{tr}\left[\left(\frac{\partial W}{\partial F}\right)^T F \dot{P}_c P_c^{-1}\right] + \dot{U}, \tag{4.7}$$

where a well known property of the trace operator, tr, has been used. Denoting

$$\Gamma_c = \dot{P}_c P_c^{-1}, \tag{4.8}$$

we observe that, given any smooth function Γ_c, eqn (4.8) can be regarded as a system of 9 non-linear O.D.E.'s which, in general, will have a unique solution P_c on a certain neighbourhood $[-\epsilon, \epsilon]$ of 0, with given initial conditions $P_c(0)$. Rewriting eqn (4.7) as

$$\dot{W} = \mathrm{tr}\left[\left(\frac{\partial W}{\partial F}\right)^T F \Gamma_c\right] + \dot{U}, \tag{4.9}$$

which is a single quasi-linear P.D.E. for W, we see that given any smooth Γ_c and U as functions of s, this last equation will, in general, have a unique solution $W(F, s)$. In view of the above remarks we conclude

Proposition 1

A necessary and sufficient condition for a hyperelastic body $\mathcal{B}$ with constitutive law

$$W = W(F, X) \tag{2.1}$$

to be smoothly uniform at X_0 along the curve

$$c : [-\epsilon, \epsilon] \to \mathcal{B}, \tag{4.1}$$

$$c(0) = X_0, \tag{4.2}$$

is that functions $\Gamma_c(s)$ and $U(s)$ can be found such that eqn (4.9) is satisfied for $s \in [-\epsilon, \epsilon]$.

Indeed, the solution of eqn (4.8) provides a smooth field of material isomorphisms along the curve.

If a body is such that at a given point X_0 it is smoothly uniform at a piece ("germ") of every smooth curve containing that point, the body is said to be curve-wise locally smooth at X_0.

5. THE ROLE OF THE ISOTROPY GROUP

As explained in Section 3, elements from the isotropy group can be used, by invoking eqn (3.3), to replace one material isomorphism with another. Such an extra degree of freedom in the choice of material isomorphisms plays a central role in smoothness considerations, particularly in the case in which the symmetry group is continuous.

For definiteness, assume that $\mathcal{G}$ is a one-parameter group† with parameter λ. From eqn (3.7) we know that the isotropy group at an arbitrary point X is given by

$$\mathcal{G}_X = P_X \mathcal{G} P_X^{-1}, \tag{5.1}$$

†Such is the case, for example, for transversely isotropic materials. For higher dimensional continuous groups similar considerations apply, by confining the attention to one-parameter subgroups.

where P_X is any material isomorphism (2.8). The r.h.s. of eqn (3.2) can be viewed then as a function of λ. Assuming, without loss of generality, that the group identity corresponds to $\lambda = 0$, and differentiating eqn (3.2) with respect to λ at $\lambda = 0$ we obtain

$$0 = \mathrm{tr}\left[\left(\frac{\partial W}{\partial F}\right)^T F P_X \frac{\mathrm{d}G}{\mathrm{d}\lambda} P_X^{-1}\right] + \mathrm{tr}\left[\left(\frac{\partial V}{\partial G}\right)^T \frac{\mathrm{d}G}{\mathrm{d}\lambda}\right]. \tag{5.2}$$

Note that we have also used the definition of the function V introduced in eqn (3.8). Equation (5.2) expresses a differential condition to be satisfied by a uniform body with a continuous one-parameter isotropy group. Comparing eqns (4.9) with (5.2) we conclude

Proposition 2

Given any functions Γ_c and U satisfying eqn (4.9) for a material with a one-parameter isotropy group, new functions satisfying eqn (4.9) can be found by the expressions

$$\hat{\Gamma}_c = \Gamma_c + \psi(s) P_c \frac{\mathrm{d}G}{\mathrm{d}\lambda} P_c^{-1}, \tag{5.3}$$

and

$$\dot{\hat{U}} = \dot{U} + \dot{\psi}(s)\, \mathrm{tr}\left[\left(\frac{\partial V}{\partial G}\right)^T \frac{\mathrm{d}G}{\mathrm{d}\lambda}\right], \tag{5.4}$$

where $\psi(s)$ is an arbitrary smooth function of s only.

A converse to eqn (5.2) can be motivated as follows. Let R_X be a tensor in $T_X\mathscr{B}$ satisfying at X

$$0 = \mathrm{tr}\left[\left(\frac{\partial W}{\partial F}\right)^T F R_X\right] + \mathrm{tr}\left[\left(\frac{\partial V}{\partial G}\right)^T_{G=1} P_X^{-1} R_X P_X\right], \tag{5.5}$$

identically for all F. This can also be written as

$$0 = \frac{\mathrm{d}}{\mathrm{d}\epsilon}[W(F\,\mathrm{e}^{\epsilon R_X}) + V(\mathrm{e}^{\epsilon R})]_{\epsilon=0} \tag{5.6}$$

with

$$R = P_X^{-1} R_X P_X. \tag{3.7}$$

Note that, since eqn (5.6) is satisfied identically in F, we conclude, using eqn (3.10) that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}[W(F\,\mathrm{e}^{\epsilon R_X}) + V(\mathrm{e}^{\epsilon R})]_{\epsilon=\epsilon_0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}[W(F\,\mathrm{e}^{\epsilon_0 R_X}\,\mathrm{e}^{(\epsilon-\epsilon_0)R_X}) + V(\mathrm{e}^{(\epsilon-\epsilon_0)R}) + V(\mathrm{e}^{\epsilon_0 R})]_{(\epsilon-\epsilon_0)=0} \tag{5.7}$$
$$= 0,$$

for all ϵ_0, and therefore eqn (5.7) is a differential equation, whose unique solution is obviously

$$W(F\,\mathrm{e}^{\epsilon R_X}) + V(\mathrm{e}^{\epsilon R}) = W(F), \tag{5.8}$$

which shows that R_X is the differential at the origin of the one-parameter subgroup $\mathrm{e}^{\epsilon R_X}$ of the isotropy group. Incidentally, this result proves that for a finite (discrete) isotropy group, there is no R_X satisfying eqn (5.5).

The functions $\psi(s)$ can be used as some kind of *analytic continuators*. For, let a curve $c(s)$ joining 2 points A, B (Fig. 2) be given and let ${}_A\Gamma_c$, ${}_AU$ and ${}_B\Gamma_c$, ${}_BU$ be functions satisfying eqn (4.9) on intervals around A and B, respectively, but not coinciding on the (non-empty) intersection of the intervals. On the intersection, however, the differences ${}_A\Gamma_c - {}_B\Gamma_c$ and

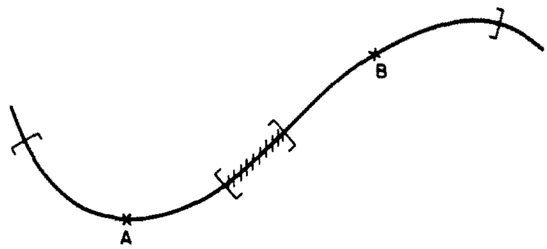

Fig. 2.

${}_A\dot{U} - {}_B\dot{U}$ satisfy certain conditions which we now investigate. To this end, we note first that, since a point must be materially isomorphic to itself, we must have at all points of the intersection

$$ {}_{BA}G = {}_BP_c^{-1}\,{}_AP_c \in \mathscr{G}, \tag{5.9}$$

where

$$ {}_A\Gamma_c = {}_A\dot{P}_c\;{}_AP_c^{-1}, \tag{5.10}$$

and similarly for ${}_BP_c$. On the other hand it follows from eqn (2.10) that

$$\hat{W}(F_AP_c) + {}_AU = \hat{W}(F_BP_c) + {}_BU, \tag{5.11}$$

so that using (5.9) and (3.8) we obtain

$$\begin{aligned}\hat{W}(F_BP_c) + {}_BU &= \hat{W}(F_BP_c\;{}_BP_c^{-1}\,{}_AP_c) + {}_BU + V({}_BP_c^{-1}\,{}_AP_c)\\ &= \hat{W}(F_AP_c) + {}_BU + V({}_BP_c^{-1}\,{}_AP_c),\end{aligned} \tag{5.12}$$

which, when compared with (5.11), yields

$$ {}_AU - {}_BU = V({}_BP_c^{-1}\,{}_AP_c) = V({}_{BA}G). \tag{5.13}$$

Differentiating (5.13) with respect to s we obtain

$$ {}_A\dot{U} - {}_B\dot{U} = \mathrm{tr}\left\{\left[\left(\frac{\partial V}{\partial G}\right)^T_{G={}_{BA}G}\right]{}_{BA}\dot{G}\right\} \tag{5.14}$$

or, using eqns (5.9) and (3.11),

$$ {}_A\dot{U} - {}_B\dot{U} = \mathrm{tr}\left[\left(\frac{\partial V}{\partial G}\right)^T_{G=1}{}_BP_c^{-1}({}_A\Gamma_c - {}_B\Gamma_c){}_BP_c\right], \tag{5.15}$$

which shows that the U's and the Γ's are connected in a peculiar way. Evaluating (4.9) for both sets of variables and subtracting yields

$$0 = \mathrm{tr}\left[\left(\frac{\partial W}{\partial F}\right)^T F({}_A\Gamma_c - {}_B\Gamma_c)\right] + {}_A\dot{U} - {}_B\dot{U}, \tag{5.16}$$

or, using (5.15),

$$0 = \mathrm{tr}\left[\left(\frac{\partial W}{\partial F}\right)^T F({}_A\Gamma_c - B\Gamma_c)\right] + \mathrm{tr}\left[\left(\frac{\partial V}{\partial G}\right)^T_{G=1}{}_B\dot{P}_c^{-1}({}_A\Gamma_c - {}_B\Gamma_c){}_BP_c\right]. \tag{5.17}$$

Comparing (5.17) with (5.5) we arrive at the

Proposition 3

If two pairs of functions $({}_A\Gamma_c, {}_AU)$ and $({}_B\Gamma_c, {}_BU)$ satisfy eqn (4.9) along a curve of a material with a one-parameter isotropy group, then there exists a function $\psi(s)$ of position along the curve such that

$$ {}_BP_c^{-1}({}_A\Gamma_c - {}_B\Gamma_c){}_BP_c = \psi\left(\frac{dC}{d\lambda}\right)_{\lambda=0}, \tag{5.18} $$

$$ {}_A\dot{U} - {}_B\dot{U} = \psi \operatorname{tr}\left[\left(\frac{\partial V}{\partial G}\right)^T_{G=1} \frac{dG}{d\lambda}\right]. \tag{5.19} $$

This last formula follows from (5.16) and from (5.2). Proposition 3 is the converse of Proposition 2, as follows from (5.18) and (5.19) in the region of overlap.

Corollary

It is always possible to extend smoothly Γ_c and U to a unique smooth expression on the two intervals shown in Fig. 2.

Indeed, it is enough to extend smoothly $\psi(s)$ to the whole interval $\mathcal{U}_B$ around B and to set

$$ \Gamma_c = \begin{cases} {}_A\Gamma_c \text{ on } \mathcal{U}_A, \\ {}_B\Gamma_c + \psi_B P_c^{-1}\left(\frac{dG}{d\lambda}\right)_{\lambda=0} {}_BP_c, \text{ on } \mathcal{U}_B - \mathcal{U}_A, \end{cases} \tag{5.20} $$

and similarly for U.

We note, incidentally, that if the symmetry group is discrete, there is only one possible Γ_c, and therefore the continuation is always trivial.

Remark

Even in the trivial case, i.e. Γ_c unique, the body may not be smoothly uniform along all of c since P_c may suffer a discontinuity at the overlap.

6. SMOOTH UNIFORMITY

In Definition 1 the dependence of the isomorphisms P_X on X was left completely arbitrary. If, however, a smooth field of P_X can be found over the entire body, we say that $\mathcal{B}$ enjoys *smooth uniformity*. It may happen that even though such a global smooth field cannot be found, a neighbourhood of each point exists such that the field P_X is smooth on it.† In this case we say that $\mathcal{B}$ enjoys *locally smooth uniformity*.

Following the treatment in Section 4 we try to obtain a differential condition for locally smooth uniformity. Differentiating eqn (2.10) with respect to F

$$ \frac{\partial W}{\partial F} = \frac{\partial \hat{W}}{\partial FP_X} P_X{}^T, \tag{6.1} $$

and with respect to X

$$ \frac{\partial W}{\partial X} = \operatorname{tr}\left[\left(\frac{\partial \hat{W}}{\partial FP_X}\right)^T F \frac{\partial P_X}{\partial X}\right] + \frac{\partial U}{\partial X}, \tag{6.2} $$

and combining the results, we obtain

$$ \frac{\partial W}{\partial X} = \operatorname{tr}\left[\left(\frac{\partial W}{\partial F}\right)^T F\Gamma\right] + U, \tag{6.3} $$

†This concept was introduced by Wang in[2], who called it simply "smooth uniformity". This situation is analogous to trying to cover the surface of a sphere with just one coordinate system with no singularities. It cannot be done. But any given point can be surrounded by a well-behaved coordinate patch.

where now

$$\mathbf{U} = \frac{\partial U}{\partial X}, \tag{6.4}$$

$$\Gamma = \frac{\partial P_X}{\partial X} P_X^{-1}. \tag{6.5}$$

Note that when written out in reference coordinates Γ will turn out to have 3 free indices.

From the preceding derivation it is clear that the existence of smooth functions $\mathbf{U}$, Γ on a neighbourhood of each point satisfying eqn (6.3) is a necessary condition for locally smooth uniformity.

On the other hand, let such functions be given satisfying eqn (6.3) and let a curve c be specified through a point. Then from the arguments of the preceding sections it follows that the body is curve-wise locally smooth. Note that since the material isomorphisms P_c may turn out to be path dependent, we cannot infer sufficiency from eqn (5.3). Because of this possible path dependence it is natural to call Γ a material connection. The answer to the question "given Γ, does there exist a smooth field of material isomorphisms P_X such that eqn (6.5) is satisfied?", is affirmative if and only if the Riemann–Christoffel tensor $R(\Gamma)$ vanishes identically on the neighbourhood† Similarly $\mathbf{U}$ must have a vanishing rotor so that it can be derived from a scalar field U.

Let a locally smooth uniform body be given and let P_X and U denote neighbourhood-wise smooth fields satisfying eqn (6.3). We wish to find whether globally smooth functions Γ and U can be constructed which satisfy eqn (6.3). To this end we note that the same arguments leading to proposition 2 will apply now and we conclude

Proposition 4

Let $\mathcal{B}$ be a locally smooth uniform body with a one-parameter (λ) symmetry group $\mathcal{G}$.

Then, if P_X and U denote the neighbourhood-wise smooth fields satisfying eqn (5.3), the functions

$$\hat{\Gamma} = \frac{\partial P_X}{\partial X} P_X^{-1} + \psi(X) P_X \frac{\mathrm{d}G}{\mathrm{d}\lambda} P_X^{-1}, \tag{6.6}$$

$$\hat{\mathbf{U}} = \frac{\partial U}{\partial X} + \psi(X)\,\mathrm{tr}\left[\left(\frac{\mathrm{d}V}{\mathrm{d}G}\right)^{T} \frac{\mathrm{d}G}{\mathrm{d}\lambda}\right], \tag{6.7}$$

where $\psi(X)$ is an arbitrary vector-valued neighbourhood-wise smooth function, also satisfy eqn (6.3).

In a similar way Proposition 3 and its corollary can be generalized. We note, finally, that in the case of a discrete isotropy group the material connection must be unique.

Remark

One can develop a criterion for uniformity entirely in terms of a given strain energy density W, eqn (2.1). This can be achieved by first differentiating the basic eqn (6.3) with respect to F. *This provides* 9 *additional vector equations which can be used to solve for the components of* Γ. The requirement that these Γ's be independent of F then provides a rather complex differential condition on W, which is necessary for uniformity. A complete development of this approach, including questions of sufficiency, integrability and uniqueness, provides a challenge for future research.

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†Note this requirement is just the integrability condition for the system of eqns (6.5).

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